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Approximate capacities of some toroidal condensers

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Abstract. It is shown that a variety of electrostatic problems involving toroidal condensers are solvable approximately by means of a certain type of transformation of Laplace's equation and perturbation theory. After a critique of previous theory of the toroidal condenser with concentric-circular section, this problem is re-solved, followed by the problem of the elliptic-toroidal condenser. On the basis of these solutions, other problems are solved approximately by a double-perturbation approach; namely, the problem of a circular torus containing a laminar ring, and that of the circular-toroidal condenser for which the circular sections are eccentric.

1. Introduction

If two-dimensional orthogonal curvilinear coordinates are defined on a region of the xy plane for, say, $y \geq 0$, and are considered to be rotated about the x axis in the sense that an azimuthal variable ϕ is added, then together with ϕ they form a three-dimensional orthogonal curvilinear coordinate system. This might be a well-known system; for example, by rotating plane polar coordinates (r, θ) defined on the whole half-plane, one obtains spherical polar coordinates (r, θ, ϕ) defined on all of space. But this is exceptional. Generally, one will obtain one of a large class of coordinate systems of which all but a few special ones are new. They are, generally speaking, of little use in mathematical physics, for in every case except where the system is a well-known one, it is defined only on a multiply connected toroidal region. However, it could be of use when the problem in hand is itself concerned with the interior of a torus.

This fact was perceived, in the context of electrostatics and in one special case, at least, by Waters (1956), who gave a theory of the toroidal condenser whose surfaces are formed by the rotation of *concentric* circles, as shown in figure 1. The plane coordinate system rotated consists of plane polar coordinates (r, θ), with pole at the point $(0, b)$ $(b > 0)$ and initial line from $(0, b)$ through the origin. They are defined for $c \leq r \leq a$ $(a < b)$, where c and a are the radii of the interior and exterior circles respectively, and for all θ reckoned clockwise from 0 to 2π. Thus, relative to the left-hand axes and ϕ direction of figure 1, the orthogonal curvilinear coordinate system of space, (r, θ, ϕ), is related to the spatial Cartesian coordinates (x, y, z) by

$$x = r \sin \theta, \qquad y = (b - r \cos \theta) \cos \phi, \qquad z = (b - r \cos \theta) \sin \phi. \tag{1}$$

In order to solve the condenser problem, Waters (1956) transforms Laplace's equation into these coordinates, whereupon, for an axisymmetric potential $V(r, \theta)$,

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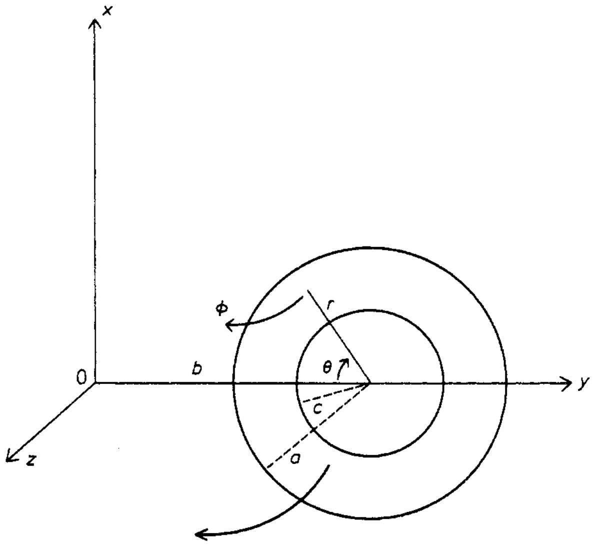

Figure 1. Concentric circles which are rotated to generate the surfaces of a toroidal condenser.

independent of ϕ, it becomes

$$\frac{\partial}{\partial r}\left[r\left(1-\frac{r}{b}\cos\theta\right)\frac{\partial V}{\partial r}\right]+\frac{1}{r}\frac{\partial}{\partial\theta}\left[\left(1-\frac{r}{b}\cos\theta\right)\frac{\partial V}{\partial\theta}\right]=0. \tag{2}$$

This equation, despite its neat and concise form, is not simple, for it is not separable, nor is any change of variable apparent under which it becomes so. Nevertheless, Waters gives an incisive theory of it, but one which, most unfortunately, appears to contain a deep-seated error. His formula for the capacity,

$$C=\frac{\pi b}{\log a/c}\left[1-\frac{a^2}{b^2}\left(1-\frac{c^2}{a^2}-\frac{2c}{a}\log\frac{a}{c}\right)\right] \tag{3}$$

(CGS electrostatic units), which he claims to be exact, is evidently invalid.

We shall explain the error and shall reach the position of doubting whether (2) is amenable to exact treatment. We shall then re-solve the problem *approximately*, by perturbation theory, regarding a/b (and thus r/b) as small and treating the difference of (2) from the plane-polar form of Laplace's equation as a perturbation upon the latter. In fact, by this method, the solution for C comes out as a power series in a/b, so that (3), if it were correct at all, would be the solution to the second order. We shall obtain the fourth-order solution.

As we have indicated, (r, θ, ϕ), related to (x, y, z) according to (1), is but one of a large class of coordinate systems obtainable by the rotation of plane systems, and after solving the problem for the condenser described above, we shall solve approximately some other toroidal condenser problems.

The classical approach to torus problems in electrostatics is the theory of toroidal harmonics (see Hobson 1965, Morse and Feshbach 1953). This leads to solutions which are exact but, even for the simplest problems, infinite series of difficult functions (the

Legendre functions of half-integral order). The real disadvantage in the present context, however, is that the only tractable condenser problem is one for which the generating circles are not concentric but, for given radii, have specific eccentricity (for they correspond to constant values of a toroidal coordinate and as such, belong to a precise coaxial, but not concentric, system). There is merit in studying the situation in which there is *arbitrary* eccentricity, and this is a problem which we shall solve approximately, but clearly, one in which there is *specific* eccentricity is of only very slight physical interest. The case of concentric generating circles is, from the physical point of view, overwhelmingly the most interesting and is the one to which we give our main attention.

2. Nomenclature

The toroidal systems we study will always be coaxial, but there will be no need to use the word 'coaxial' as nothing seems to be known about toroidal electrostatic systems which do not have this property.

Thus a torus is always a surface of revolution, and we describe it according to the form of the generating curve. For example, the rotation of a circle gives what we call a circular torus. If the two conducting surfaces of a toroidal condenser are generated by curves of the same description, we use this description for the condenser itself. For example, the condensers described in the last section are circular-toroidal ones. To refine the description, we must speak of the relative quality of the generating curves. Thus we may have a concentric-section circular-toroidal condenser, or an eccentric-section circular-toroidal condenser, or, as we shall see later, a confocal-section elliptic-toroidal condenser.

The above nomenclature may still not describe a situation adequately since, for example, an eccentric-section circular-toroidal condenser is not necessarily such that one torus is inside the other. However, we would refer to one in which the tori were mutually outside as a non-enclosed circular-toroidal condenser, although this will not concern us in this paper. If the surfaces of a condenser are generated by curves of different form, no concise nomenclature seems possible.

The above condensers are fundamentally different from the one studied by Bolduc *et al* (1973), in which the two conducting surfaces are parts of the same torus, split, so to speak, and to which they refer as the 'the' toroidal condenser.

3. The concentric-section circular-toroidal condenser: critique of previous theory

Recognising that equation (2) is not separable, Waters (1956) seeks a solution in the form of a Fourier series:

$$V=\sum_{n=0}^{\infty} a_n(\mu)\cos n\theta, \tag{4}$$

where μ is the dimensionless coordinate r/b. Substitution into (2) and use of the orthogonality of the cosines gives

$$\mu\frac{\mathrm{d}^2a_0}{\mathrm{d}\mu^2}+\frac{\mathrm{d}a_0}{\mathrm{d}\mu}-\mu\frac{\mathrm{d}a_1}{\mathrm{d}\mu}-\frac{1}{2}\mu^2\frac{\mathrm{d}^2a_1}{\mathrm{d}\mu^2}=0,$$

$$\mu\frac{d^2a_1}{d\mu^2}+\frac{da_1}{d\mu}-\frac{a_1}{\mu}-\mu\frac{da_0}{d\mu}-\frac{1}{2}\mu^2\frac{d^2a_0}{d\mu^2}+a_2-\mu\frac{da_2}{d\mu}-\frac{1}{2}\mu^2\frac{d^2a_2}{d\mu^2}=0,$$

$$\mu\frac{d^2a_n}{d\mu^2}+\frac{da_n}{d\mu}-\frac{n^2}{\mu}a_n+\frac{1}{2}n(n-1)a_{n-1}-\mu\frac{da_{n-1}}{d\mu}-\frac{1}{2}\mu^2\frac{d^2a_{n-1}}{d\mu^2}$$
$$+\frac{1}{2}n(n+1)a_{n+1}-\mu\frac{da_{n+1}}{d\mu}-\frac{1}{2}\mu^2\frac{d^2a_{n+1}}{d\mu^2}=0 \qquad (n>1). \tag{5}$$

The problem is to determine the a_n recursively, which will evidently be possible if one can find a_0. Now by (4) and orthogonality again, a_0 must carry the boundary conditions, which are that V has given distinct constant values when $\mu = c/b$ and a/b, for the other coefficients by necessity vanish when μ has these values. Waters, noting that the simple coaxial-cylinder solution for a_0 will not do, since it will not allow a_1, as subsequently found, to vanish when μ is in turn both c/b and a/b, guesses a slightly less simple form for a_0 for which this obstacle does not occur, and then finds, indeed, that he can determine all of the a_n.

The difficulty is that one can make other guesses for a_0 which serve equally well, in fact, infinitely many, so that one can obtain infinitely many distinct 'solutions'. This is of course impossible by uniqueness—and it is uniqueness which is the pitfall of Waters' argument, for having obtained one procedure which seems to work, one is impelled to invoke it immediately to infer that the result is correct.

We are now confronted with the question of what is the feature which makes all but one of the infinity of 'solutions' incorrect. We can only answer this by default, as it were, looking for something in the theory which is not assuredly sound and inferring that it must have gone wrong. The striking possibility seems to be convergence. Waters does not show that the series he obtains, for example, is convergent, nor, since it contains both positive and negative powers of μ, is it by any means obviously so. We conclude that its status is no higher than that of any other of the infinitely many possibilities, and that the *a priori* likelihood of its being the solution is nil.

Evidently the only hope of making the method work is to approach equations (5) deductively, attempting, say, a successive elimination of the a_n for $n > 0$, with a view to obtaining by a limiting process an equation for a_0 alone. This is a most unpromising undertaking, and efforts of the kind by the present author have been fruitless. The problem is essentially a case of the general one of solving an infinite set of linear simultaneous differential equations, about which it seems safe to say that little is known, especially when, as here, the coefficients are non-constant.

4. The concentric-section circular-toroidal condenser: approximate new theory

The approximate method we use is a straightforward application of standard perturbation theory, and it will not therefore need a great deal of description.

In the limit as $b \to \infty$, (2) becomes the plane-polar form of Laplace's equation, appropriate to the limiting situation in which the toroidal condenser becomes a coaxial-cylindrical condenser. The solution for the potential in this case, having the value V_c ($\neq 0$) when $r = c$, and 0 when $r = a$, is well known to be

$$V_0 = \frac{V_c}{\log a/c}\log\frac{a}{r}. \tag{6}$$

Now let us express r/b in (2) as $(r/a)(a/b)$ and agree to regard a/b as small. Then if we write $V_1 = V_0 + (a/b)\psi_1(r, \theta)$, substitute this into (2) and drop terms with the factor a^2/b^2 as they occur, we find that ψ_1 must be of the form $\lambda_{11}(r)\cos\theta$ and obtain a second-order ordinary differential equation for λ_{11} which is easily solvable by elementary methods. Solution introduces two constants, and these we choose so that $\lambda_{11}(c) = \lambda_{11}(a) = 0$, for then we maintain the boundary conditions by having $V_1(c) = V_c$, $V_1(a) = 0$.

This is the first stage in the perturbation process. Now, knowing both V_0 and ψ_1, we write $V_2 = V_0 + (a/b)\psi_1(r, \theta) + (a^2/b^2)\psi_2(r, \theta)$, substitute into (2) and drop terms with a^3/b^3. We hence find that ψ_2 must be of the form $\lambda_{20}(r) + \lambda_{22}(r)\cos 2\theta$, and obtain independent second-order ordinary differential equations for λ_{20} and λ_{22} which again are easily solvable. Choosing the constants so that $\lambda_{20}(c) = \lambda_{20}(a) = 0$, $\lambda_{22}(c) = \lambda_{22}(a) = 0$, we shall now have V_2 satisfying the boundary conditions.

The process we are describing can be carried to any order, and seems never to be more than a matter of elementary algebra and calculus, although at the third order the algebra becomes heavy and will eventually be so to the extent of intractability. In fact, what we are finding at the nth order is the nth term of a series solution of the form

$$\begin{aligned} V &= V_0 + \sum_{n=1}^{\infty} \frac{a^n}{b^n}\psi_n(r, \theta), \\ \psi_{2k-e} &= \sum_{h=e}^{k} \lambda_{2k-e,2h-e}(r)\cos(2h-e)\theta, \qquad e = 0, 1. \end{aligned} \tag{7}$$

It is characteristic of perturbation theory, that we can never know this series completely, but can in principle calculate its nth partial sum V_n for n as large as we please.

In terms of the potential V, the surface charge density on the inner torus is, by Coulomb's theorem, $-(1/4\pi)(\partial V/\partial r)_{r=c}$, and the surface element is given by $c(b - c\cos\theta)\,\mathrm{d}\theta\,\mathrm{d}\phi$. Hence the capacity of the toroidal condenser in which the outer torus is the zero-potential ('grounded') conductor is

$$C = \frac{-c}{4\pi V_c}\int_0^{2\pi}\int_0^{2\pi}\left(\frac{\partial V}{\partial r}\right)_{r=c}(b - c\cos\theta)\,\mathrm{d}\theta\,\mathrm{d}\phi, \tag{8}$$

and of course, the ϕ integration can be carried out at once to give a factor 2π. More particularly, the $2n$th approximation to C is found, with the use of (6) and (7), to be

$$C_{2n} = \frac{\pi b}{\log a/c} - \frac{\pi bc}{V_c}\sum_{k=1}^{n}\frac{a^{2k}}{b^{2k}}\left[\left(\frac{\mathrm{d}\lambda_{2k,0}}{\mathrm{d}r}\right)_{r=c} - \frac{c}{2a}\left(\frac{\mathrm{d}\lambda_{2k-1,1}}{\mathrm{d}r}\right)_{r=c}\right], \tag{9}$$

the capacity being modified only at even orders of the perturbation parameter (here a/b), as always seems to happen in perturbation theory, although we know of no general theorem to this effect.

As we have said, the detail of the calculation is elementary, if in its later stages tedious, and a closer description is hardly called for. The result for the capacity to the fourth order is

$$\begin{aligned} C_4 = \frac{\pi b}{\log a/c}\Bigg[1 &+ \frac{a^2}{8b^2}\left(\frac{2c^2\log a/c}{a^2-c^2} - \frac{a^2-c^2}{2a^2\log a/c}\right) \\ &- \frac{a^4}{128b^4}\left(\frac{4c^4\log^2 a/c}{(a^2-c^2)^2} - \frac{c^4\log a/c}{a^4-c^4} - \frac{6c^2}{a^2}\right. \end{aligned}$$

$$+\frac{29}{16}\frac{a^4-c^4}{a^4\log a/c}-\frac{(a^2-c^2)^2}{2a^4\log^2 a/c}\Big)\Big]. \tag{10}$$

This is a clear negation of Waters' claim that a result to the order a^2/b^2 is exact, and moreover, we see how completely different is the a^2/b^2 term from the corresponding one in his formula, (3).

We write down the result for the *third*-order field, as this might be of some interest in itself:

$$\begin{aligned}
V_3=\frac{V_c}{\log c/a}\Big[\!\Big[&\log\frac{r}{a}-\frac{1}{b}\Big[\Big(\frac{1}{2}\log r-\frac{1}{4}+\frac{\alpha_1}{2}\Big)r+\frac{\beta_1}{r}\Big]\cos\theta\\
&-\frac{1}{b^2}\Big[\frac{1}{8}r^2\log r+\frac{1}{8}\alpha_1 r^2+\alpha_2\log r+\beta_2\\
&+\Big(\frac{3}{16}r^2\log r+\frac{\beta_1}{4}+\alpha_3 r^2+\frac{\beta_3}{r^2}\Big)\cos 2\theta\Big]\\
&-\frac{1}{b^3}\Big\{\Big[\frac{9}{64}r^3\log r+\Big(\frac{1}{32}+\frac{3\alpha_1}{32}+\frac{\alpha_3}{4}\Big)r^3\\
&+\Big(\frac{\alpha_2}{2}-\frac{\beta_1}{8}\Big)r\log r+\alpha_4 r+\frac{\beta_4}{r}\Big]\cos\theta\\
&+\Big(\frac{5}{64}r^3\log r+\frac{\beta_3}{4r}+\frac{3}{32}\beta_1 r+\alpha_5 r^3+\frac{\beta_5}{r^3}\Big)\cos 3\theta\Big\}\Big]\!\Big],
\end{aligned}$$

$$\begin{aligned}
&\alpha_1=\frac{1}{2}-\frac{a^2\log a-c^2\log c}{a^2-c^2}, \qquad \beta_1=\frac{1}{2}\frac{a^2c^2}{a^2-c^2}\log\frac{a}{c},\\
&\alpha_2=\frac{-(a^2-c^2)}{16\log a/c}, \qquad \beta_2=\frac{1}{16}\Big(\frac{2a^2c^2\log a/c}{a^2-c^2}+\frac{a^2\log c-c^2\log a}{\log a/c}\Big),\\
&\alpha_3=-\frac{3a^4\log a+2a^2c^2\log a/c-3c^4\log c}{16(a^4-c^4)}, \qquad \beta_3=\frac{1}{16}\frac{a^4c^4}{a^4-c^4}\log\frac{a}{c},\\
&\alpha_4=\frac{a^2c^2\log a/c}{8(a^2-c^2)}-\frac{5}{64}(a^2+c^2)+\frac{a^2\log a-c^2\log c}{16}\Big(\frac{1}{2\log a/c}+\frac{a^2c^2\log a/c}{(a^2-c^2)^2}\Big),\\
&\beta_4=\frac{3}{64}a^2c^2+\frac{a^4c^4}{64(a^4-c^4)}\log\frac{a}{c}-\frac{a^4c^4}{16(a^2-c^2)^2}\log^2\frac{a}{c},\\
&\alpha_5=\frac{-5}{64}\Big(\frac{a^6\log a-c^6\log c}{a^6-c^6}\Big)-\frac{a^2c^2(3a^4+7a^2c^2+3c^4)\log a/c}{64(a^2+c^2)(a^6-c^6)},\\
&\beta_5=\frac{1}{64}\frac{a^6c^6}{a^6-c^6}\log\frac{a}{c}.
\end{aligned} \tag{11}$$

We did not calculate the field fully to the fourth order, for the algebra is very heavy at this stage, while, as we see by (9), only the λ_{40} part of the a^4/b^4 term is needed for the calculation of the fourth-order capacity, (10). Of course, while the difference $C-C_4$ (the error of our result, as it were) is of order a^6/b^6, the difference $V-V_3$ is of order a^4/b^4.

5. The confocal-section elliptic-toroidal condenser

Elliptic coordinates (ξ, η) are defined by their relationship to Cartesian coordinates, which we shall denote by (x'', y''), by

$$x'' = f \sinh \eta \sin \xi, \qquad y'' = f \cosh \eta \cos \xi, \tag{12}$$

where f is a constant with the property that the (Cartesian) points $(0, f)$, $(0, -f)$ are the common foci of the family of ellipses given by constant values of the coordinate η. If we make the Cartesian transformation

$$x' = x'' \cos \alpha + y'' \sin \alpha, \qquad y' = y'' \cos \alpha - x'' \sin \alpha, \tag{13}$$

a rotation, then the major axis of each ellipse makes the angle α with the y' axis. We next make a translation, introducing coordinates (x, y) by

$$x = x', \qquad y = y' + b \ (b > 0), \tag{14}$$

so that, while the major axes make the angle α with the y axis, the centres of the ellipses are the point $(0, b)$. The disposition of any particular ellipse in the respective Cartesian systems is illustrated in figure 2(a)–(c).

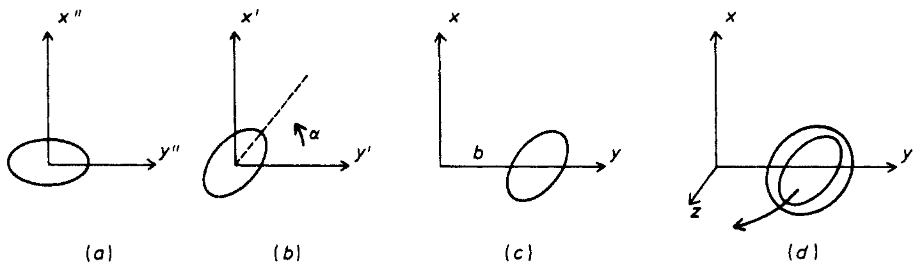

Figure 2. An ellipse relative to successive sets of coordinate axes; two confocal ellipses in the final set.

We shall be concerned with the region of the plane bounded by the ellipses $\eta = \eta_0$, $\eta = \eta_1 (> \eta_0)$, that is, for $\eta_0 \leq \eta \leq \eta_1$, and we shall suppose that b in (14) is large enough for the ellipse $\eta = \eta_1$, which encloses the ellipse $\eta = \eta_0$, to be entirely to the right of the x axis (figure 2(d)). If we now 'rotate' the coordinates (ξ, η), $\eta_0 \leq \eta \leq \eta_1$, about the x axis in the manner described in § 1, we shall have an orthogonal curvilinear coordinate system for space, (ξ, η, ϕ), related to the left-hand Cartesian coordinates (x, y, z) by

$$\begin{aligned} x &= f \sinh \eta \sin \xi \cos \alpha + f \cosh \eta \cos \xi \sin \alpha, \\ y &= (f \cosh \eta \cos \xi \cos \alpha - f \sinh \eta \sin \xi \sin \alpha + b) \cos \phi, \\ z &= (f \cosh \eta \cos \xi \cos \alpha - f \sinh \eta \sin \xi \sin \alpha + b) \sin \phi, \end{aligned} \tag{15}$$

and these coordinates will be suitable for describing the elliptic-toroidal region generated by rotation of the plane region given by $\eta_0 \leq \eta \leq \eta_1$.

We shall regard the tori $\eta = \eta_0$, $\eta = \eta_1$ as the conducting surfaces of a confocal-section elliptic-toroidal condenser in which the outer conductor is grounded, and, with a view to finding the capacity of this condenser, shall transform Laplace's equation into the coordinates (ξ, η, ϕ). This is easily done by use of a standard formula (holding for curvilinear coordinates generally, see Hobson (1965) or Jeans (1960)), and the result for an axisymmetric potential $V(\xi, \eta)$, expressed in a form convenient for our purposes, is

$$\frac{\partial^2 V}{\partial \xi^2}+\frac{\partial^2 V}{\partial \eta^2}-\frac{f(\cosh \eta \sin \xi \cos \alpha+\sinh \eta \cos \xi \sin \alpha)}{b+f(\cosh \eta \cos \xi \cos \alpha-\sinh \eta \sin \xi \sin \alpha)} \frac{\partial V}{\partial \xi}$$
$$+\frac{f(\sinh \eta \cos \xi \cos \alpha-\cosh \eta \sin \xi \sin \alpha)}{b+f(\cosh \eta \cos \xi \cos \alpha-\sinh \eta \sin \xi \sin \alpha)} \frac{\partial V}{\partial \eta}=0. \tag{16}$$

It is clear that, without the first-derivative terms, (16) is the elliptic-coordinate form of Laplace's equation, and that, regarding $\cosh \eta/b$, $\sinh \eta/b$ as small, the extra terms can be treated as a perturbation in the same manner as in the last section in the case of equation (2). The unperturbed solution, having value V_c $(\neq 0)$ on the inner torus and vanishing on the outer one, is

$$V_0=V_c\left(\frac{\eta_1-\eta}{\eta_1-\eta_0}\right), \tag{17}$$

and a first-order solution of the form $V_1=V_0+(f/b)\psi_1(\xi, \eta)$, where $\psi_1=\lambda_{11}(\eta) \times\cos \xi+\hat{\lambda}_{11}(\eta) \sin \xi$, is found to fit; then a second-order solution of the form $V_2=V_0+(f/b)\psi_1(\xi, \eta)+(f^2/b^2)\psi_2(\xi, \eta)$, where $\psi_2=\lambda_{20}(\eta)+\lambda_{22}(\eta) \cos 2\xi +\tilde{\lambda}_{22}(\eta) \sin 2\xi$, and so on. In fact, the procedure is very like the one in the last section and not in the least more difficult. Conspicuous differences are, firstly, the occurrence of sines as well as cosines, due to the present system not having, in general, symmetry about an equatorial plane, and secondly, that the calculation, for orders of perturbation above the first, requires expansion of the denominator in (16) as a geometric series and subsequent truncation to the required order.

The general formula for the capacity, corresponding to (8), is now

$$C=\frac{-1}{4\pi V_c} \int_0^{2\pi} \int_0^{2\pi}\left(\frac{\partial V}{\partial \eta}\right)_{\eta=\eta_0}(b+f \cosh \eta \cos \xi \cos \alpha-f \sinh \eta \sin \xi \sin \alpha) \, d\xi \, d\phi, \tag{18}$$

and we easily find a formula corresponding to (9). However, in view of the parallelism with the previous problem, no more description than we have given seems necessary, and we give the result for the capacity to the second order:

$$C_2=\frac{\pi b}{\eta_1-\eta_0}\left\{1+\frac{f^2}{16 b^2}\left[(\eta_1-\eta_0)\left(\frac{\cosh(\eta_1+\eta_0)}{\sinh(\eta_1-\eta_0)}+\coth(\eta_1-\eta_0) \cos 2\alpha\right)\right.\right.$$
$$\left.\left.-\cos 2\alpha-\frac{1}{2} \frac{\sinh 2\eta_1-\sinh 2\eta_0}{\eta_1-\eta_0}\right]\right\}. \tag{19}$$

In terms of the lengths of the semi-major axes of the respective inner and outer elliptic sections, c and a, we have $\cosh \eta_0=c/f$, $\cosh \eta_1=a/f$ (the semi-minor axis is of course fixed by the semi-major axis and f), whereby we can express (19) in a form which, if less simple, is in terms of more tangible geometrical parameters:

$$C_2 = \frac{\pi b}{\cosh^{-1} a/f - \cosh^{-1} c/f}\left\{1 + \frac{f^2}{8b^2}\left[\left(\cosh^{-1}\frac{a}{f} - \cosh^{-1}\frac{c}{f}\right)\right.\right.$$
$$\times\left(\frac{ac\cos^2\alpha + (a^2-f^2)^{1/2}(c^2-f^2)^{1/2}\sin^2\alpha}{c(a^2-f^2)^{1/2} - a(c^2-f^2)^{1/2}}\right) - \frac{1}{2}\cos 2\alpha$$
$$\left.\left. - \frac{a(a^2-f^2)^{1/2} - c(c^2-f^2)^{1/2}}{2f^2(\cosh^{-1} a/f - \cosh^{-1} c/f)}\right]\right\}. \tag{20}$$

If a and c are fixed and f goes to zero, the elliptic sections become, in the limit, concentric circular sections with radii a and c. In this limit, therefore, (20) should become (10) truncated at the second order, and this is easily found to be the case. We therefore have an independent mutual check as far as the second order is concerned.

We note that in (19) and (20), f/b is appearing as the natural perturbation parameter. In fact, the condition $a < b$ implies that $f < eb$, where e is the eccentricity of the outer elliptic section. Therefore the need for the perturbation parameter to be 'small' is not to be understood, as it was in the previous problem, as a requirement relative to unity, but rather to e.

6. The circular-torus and enclosed laminar-ring condenser

It might reasonably be said that the problem of the last section is more interesting mathematically than physically, the elliptic-toroidal configuration being in practice an unlikely one. However, it provides the basis for the approximate solution of a problem which might be considered to be of some interest in the physical context.

In the limit $\eta_0 \to 0$, the inner elliptic section degenerates into a line segment of length $2c$, while, if η_1 is large, the outer section approximates to a circle. Then, indeed, if we replace the outer ellipse by the approximating circle, the surfaces in three dimensions correspond to a condenser whose inner conductor is a laminar ring and whose outer conductor is a circular torus (see figure 3). If $\alpha = 0$, the lamina is a so-called flat ring or

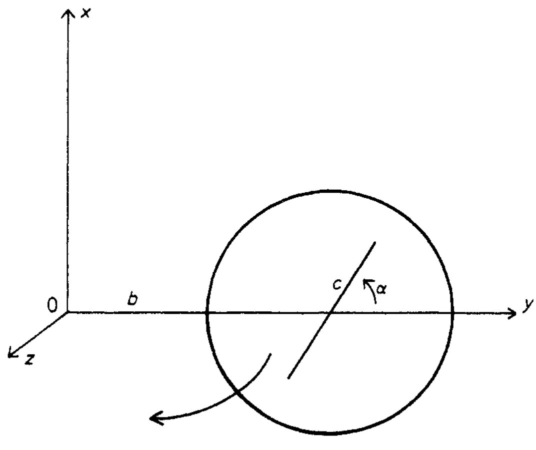

Figure 3. A circle and line segment which are rotated to generate a torus containing a laminar ring.

annular disc, while if $\alpha = \frac{1}{2}\pi$, we call it a cylindrical ring. In general, when α is neither 0 nor $\frac{1}{2}\pi$, we call it a conical ring.

The passage from the solution of an electrostatic problem in which a conducting surface is a coordinate surface of the system in use, to an approximate solution for a conducting surface which is a little different and not coincident with a coordinate surface, is made by a well-known method (Jeans 1960). It involves perturbation of the boundary condition rather than of the equation to be solved, so that in the present case, when we are already perturbing a standard form of Laplace's equation, we shall have a *double* perturbation process.

The approximating circle we take to be the one with the same centre as that of the ellipse $\eta = \eta_1$, and touching this ellipse at the ends of the major axis; that is, the one known as the director circle. Having the radius $a = f \cosh \eta_1$, it satisfies, by (12), the equation

$$\cosh^2 \eta \cos^2 \xi + \sinh^2 \eta \sin^2 \xi = \cosh^2 \eta_1,$$

the solution of which for η,

$$\eta = \cosh^{-1}[(\cosh^2 \eta_1 + \sin^2 \xi)^{1/2}], \tag{21}$$

is its equation in the form of η as a function of ξ. With the understanding that η_1 is large, $\sin \xi/\cosh \eta_1$ is small, and the Taylor expansion of (21) up to the first power in $\sin^2 \xi/\cosh^2 \eta_1$ gives us

$$\eta = \eta_1 + \frac{1}{2 \sinh 2\eta_1} - \frac{\cos 2\xi}{2 \sinh 2\eta_1}. \tag{22}$$

This is the kind of expression we want, for it is a finite trigonometrical series in terms of ξ, ending in 2ξ, exactly as is the solution for the second-order field in the problem of the last section. It provides a perturbation of the outer boundary condition which demands only a re-determination of the constants occurring in the solution of that problem.

In fact, the unperturbed solution $V_0 = A + B\eta$ must satisfy, on the inner and outer boundaries respectively,

$$A + B\eta_0 = V_c, \tag{23}$$

$$A + B\left(\eta_1 + \frac{1}{2 \sinh 2\eta_1} - \frac{\cos 2\xi}{2 \sinh 2\eta_1}\right) = 0.$$

But the boundary conditions will be satisfied by taking

$$A + B\eta_0 = V_c, \qquad A + B\eta_1 = 0,$$

which gives us again (17), and, at the outer boundary, equating to zero the sum of the remaining part of (23) and the quantity $(f^2/b^2)(\lambda_{20}(\eta_1) + \lambda_{22}(\eta_1) \cos 2\xi)$ which comes from the second-order field V_2.

The result will be a self-consistent determination of the second-order field *provided* we are correct at the outer boundary in using, instead of η as given by (22), $\eta = \eta_1$ for $\lambda_{20}(\eta)$, $\lambda_{22}(\eta)$ and $\tilde{\lambda}_{22}(\eta)$, and for $\lambda_{11}(\eta)$ and $\tilde{\lambda}_{11}(\eta)$ of the first-order field. The correctness in fact depends upon how we match orders of approximation, appreciating the double nature of the perturbation process. It is natural to look upon (22) as an expression for the second approximation in which the first-order correction is zero, in view of the correspondence of form with the second-order solution of Laplace's equation, but this is justified only if the respective second-order corrections are of comparable magnitude. If we assume that this is the case, then the use of (22) instead of $\eta=\eta_1$ in λ_{11}, $\tilde{\lambda}_{11}$, $\lambda_{20}, \ldots$ will affect only the third and fourth orders and so be superfluous. We are not tied to this assumption, however, and the conditions of a given physical situation might require otherwise; but it is the simplest we can make in an expository discussion.

In this way, we have only to re-determine $\lambda_{20}(\eta)$ and $\lambda_{22}(\eta)$ as described above. This in turn, in the case of λ_{20}, affects the determination of the capacity, the result for which to the second order (having taken $\eta_0=0$ (so that $f=c$) and $\cosh\eta_1=a/c$) is

$$C_2=\frac{\pi b}{\cosh^{-1}a/c}\left[1-\frac{c^2}{4a(a^2-c^2)^{1/2}\cosh^{-1}a/c}\right.$$
$$\left.+\frac{c^2}{16b^2}\left(\frac{2a}{(a^2-c^2)^{1/2}}\cosh^{-1}a/c\cos^2\alpha-\cos 2\alpha-\frac{a(a^2-c^2)^{1/2}}{c^2\cosh^{-1}a/c}\right)\right]. \tag{24}$$

As can be seen by comparing this result with (20) in which we put $f=c$, it is the second term in the square brackets which represents the departure from the elliptic of the circular outer section.

7. The eccentric-section circular-toroidal condenser

Our initial study of the problem of the present section will show how our general method is capable of failure.

The appropriate coordinate system for studying the region bounded by two eccentric circles is bipolar coordinates (ξ, η) related to Cartesian coordinates (X, Y) by

$$X=\frac{f\sin\xi}{\cosh\eta-\cos\xi}, \qquad Y=\frac{f\sinh\eta}{\cosh\eta-\cos\xi} \tag{25}$$

(see Morse and Feshbach 1953). In fact, the curves η = constant are a family of coaxial circles with centres on the Y axis, and if we are given arbitrarily any two circles, one inside the other, the quantitative information being their radii and the separation of their centres, it is possible to choose η_0, η_1 and f so that, according to (25), $\eta=\eta_0$ is the inner circle and $\eta=\eta_1$ the outer one. Then the coordinates (ξ, η) will describe the ring shaped region bounded by these circles.

However, we shall make a transformation similar to (12)–(14), except for there being an additional step, the first, in which we make the translation which places the origin at the centre of the inner circle. Then, calling the new Cartesian coordinates (x'', y''), we make precisely the transformations (13) and (14), choosing b large enough so that finally the x axis is outside both circles. The process is illustrated in figure 4. We now 'rotate' the coordinates (ξ, η) about the x axis in the way that led from (14) to (15), so obtaining orthogonal curvilinear coordinates of space, (ξ, η, ϕ), which describe a given eccentric-section circular-toroidal region of which the inner and outer tori are given respectively by $\eta=\eta_0$, $\eta=\eta_1$. Without the preliminary Cartesian transformation (or, as we might say, if the Cartesian transformation were the identical one, so that $x=X$, $y=Y$), (ξ, η, ϕ) are ordinary toroidal coordinates (cf Morse and Feshbach 1953, Hobson 1965).

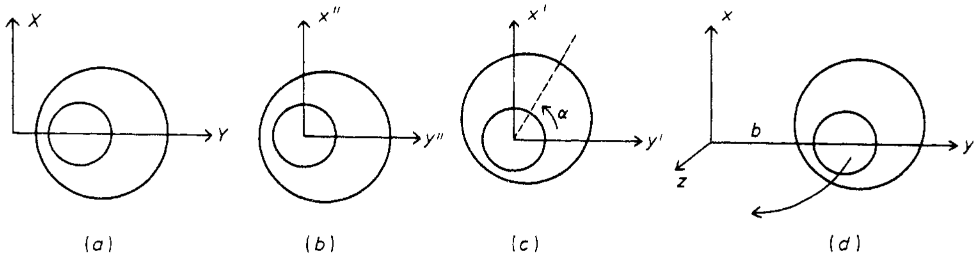

Figure 4. Two eccentric circles relative to successive sets of coordinate axes.

Transformation of Laplace's equation follows the same lines as the derivation of the form (16) from (15). We give no details and merely write down the result, again for the axisymmetric case $V = V(\xi, \eta)$:

$$\partial^2 V/\partial \xi^2 + \partial^2 V/\partial \eta^2 + D^{-1}(\cosh \eta \cos \xi \sin \alpha - \sinh \eta \sin \xi \cos \alpha - \sin \alpha)\partial V/\partial \xi$$
$$+ D^{-1}(\cos \alpha - \cosh \eta \cos \xi \cos \alpha - \sinh \eta \sin \xi \sin \alpha)\partial V/\partial \eta = 0, \qquad (26)$$
$$D = (\cosh \eta - \cos \xi)\{\sin \xi \sin \alpha + \sinh \eta \cos \alpha + [(b - d \cos \alpha)/f](\cosh \eta - \cos \xi)\}.$$

Here $d = (f^2 + c^2)^{1/2}$, where c is the radius of the inner circular section; in fact $(0, d)$ is the centre of this circle in the original Cartesian system (X, Y) (see figure 4(a)), while $(0, f)$ is the limiting position in these coordinates, of the centres of the circles $\eta =$ constant as $\eta \to \infty$.

A quick comparison of (26) with (16) suggests that the same sort of perturbation procedure should work, starting with the same zero-order solution, $V = A + B\eta$, and using now $f/(b - d \cos \alpha)$ as perturbation parameter. However, we run into difficulty, for, whereas with the successive approximation procedure for (16) the function part of the denominator disappears through the geometric series expansion, in (26) we cannot so get rid of a denominator which contains a function of ξ. The result is that, at each stage, the factor of the power of the perturbation parameter is at best an *infinite* Fourier series, assuming that we have a means of expanding the denominator in this way, and the viability of our previous perturbation analysis depended upon our having *finite* Fourier series. The situation might offer hope if we could find the explicit nature of the Fourier series, but we cannot even do this.

There are still more difficulties. Suppose that we could solve the problem. Then if the radii a and c of the respective outer and inner sections are held constant but the separation of their centres tends to zero, we should expect to be able to recover previous results obtained by using the coordinates (r, θ, ϕ) and equation (2). However, it follows by the algebra associated with (25) that, as the separation tends to zero, d tends to infinity, so that we have trouble with the perturbation parameter $f/(b - d \cos \alpha)$. It appears that, in the case of the bipolar coordinates (25), our spatial curvilinear coordinates and perturbation method are fundamentally incompatible.

In order to obtain a result at all for the problem in hand, we shall go back to (2) and modify the analysis of § 4. In fact, we invoke precisely a *double* perturbation procedure comparable with that used in the last section.

In the coordinates (x', y'), with origin the centre of the inner circle and such that the line of centres has inclination α to the x axis (see figure 4(c)), the equation of the outer circle, when we denote the separation of the respective centres by $a\epsilon$, is

$$(x' - a\epsilon \sin \alpha)^2 + (y' - a\epsilon \cos \alpha)^2 = a^2.$$

We regard ϵ as small, being our second, independent, perturbation parameter. Then introducing plane polar coordinates (r, θ) by $x' = r \sin \theta$, $y' = -r \cos \theta$ (cf § 1), and solving for r as far as ϵ^2, we obtain

$$r = a[1 + \epsilon(\cos \alpha \cos \theta - \sin \alpha \sin \theta) - \tfrac{1}{4}\epsilon^2(1 - \cos 2\alpha \cos 2\theta + \sin 2\alpha \sin 2\theta)]. \tag{27}$$

We have thus carried out a well-known procedure (cf Jeans 1960) of perturbing the outer boundary by merely (approximately) translating it.

Now, exactly as in the problem of the last section, we regard orders of approximation in the respective perturbation parameters as matching, and replace the boundary condition that the potential should vanish when $r = a$, by the condition that it should vanish when r is given by (27). We repeat the analysis of § 4, but only as far as the second order and with the present modification, noting that, having in general lost symmetry about the plane $x = 0$, the first-order field has to include a term $(a/b\tilde{\lambda}_{11}(r) \sin \theta$. In fact, because of the presence of a first-order term in (27), the calculation is more complicated than that occurring in the last section, and considerable care is necessary, although being in terms of r and θ, it is of a more familiar kind. The result for the capacity to the second order is

$$C_2 = \frac{\pi b}{\log a/c}\left[1 + \frac{a^2}{8b^2}\left(\frac{2c^2 \log a/c}{a^2 - c^2} - \frac{a^2 - c^2}{2a^2 \log a/c}\right) - \frac{a\epsilon}{b}\frac{c^2 \cos \alpha}{a^2 - c^2} + \frac{\epsilon^2 a^2}{(a^2 - c^2) \log a/c}\right]. \tag{28}$$

If we put $\epsilon = 0$, this result reduces to (10) for the concentric-section condenser, truncated at the second order. If we divide by $2\pi b$ and make $b \to \infty$ (i.e. $a/b \to 0$), we obtain a formula for the capacity per unit length of an enclosed biaxial-cylindrical condenser. In this limit when also $\epsilon = 0$, the formula is the well-known elementary one for the coaxial-cylindrical condenser. If $\alpha = \pi$ and $b = [a^2(1 - \epsilon^2) - c^2]/2a\epsilon$, the two tori are coordinate surfaces of a system of ordinary toroidal coordinates, and for this condenser the exact capacity (in infinite terms) is known by toroidal harmonics (Buchholz 1957). Thus we might expect (28) in this case to be some approximation to that result, but we have no means of making a comparison.

The first-order term in (27) leads to a coupling of the two perturbation processes, something which, at the second order, does not occur in the problem of the last section. This is represented in (28) by the term with 'mixed' factor $a\epsilon/b$ (a/b and ϵ being the two perturbation parameters). While the angle α occurs in this term, it does not in the ϵ^2 term, as must clearly be the case, for this term is still present in the above-mentioned limit as $b \to \infty$, when the condenser is a biaxial-cylindrical one whose intrinsic geometry is independent of α.

In formula (10) for the concentric-section condenser, although $C_4 \to \infty$ as $c \to a$, as it presumably must, the factor in square brackets tends to a finite limit. This is not so now, in the case of (28), but then it is not meaningful to take this limit without making $\epsilon \to 0$ correspondingly. There seems to be a hint that (28) would not be too reliable if ϵa were at all a large factor of $a - c$.

8. General discussion

In none of our solutions did we show that our perturbation process was leading to a convergent series, and it might therefore be said that our method is open to the same criticism as we ourselves made of the method of Waters (1956). However, there is a difference. The results to which the perturbation theory leads are *unique within that theory*, whereas with Waters' method this is not the case. The situation is not so much that, in the failure to find a positive explanation for the lack of uniqueness, we blamed it on non-convergence, as of the non-uniqueness being a genuine symptom of a convergence difficulty.

Referring to our critique in § 3 and then to the theory of § 4, we are able to pick out the true form of the function $a_0(\mu)$ in (4) and (5). It is, from (6) and (7),

$$a_0(\mu)=\frac{V_0}{\log a/c}\log\frac{a}{r}+\sum_{n=1}^{\infty}\frac{a^{2n}}{b^{2n}}\lambda_{2n,0}(r) \qquad (\mu=r/b), \tag{29}$$

and noting the trend of (11) (in which λ_{20} is explicitly represented), one is left in no doubt that it is a highly complicated function, the sum function of an infinite series of functions which are themselves progressively more complicated. This seems to establish conclusively the error of Waters' theory and also to confirm the suspicion we expressed, that a correct theory on the same lines is not feasible.

The theory of this paper has been essentially a perturbation theory of standard forms of Laplace's equation. Another perturbation theory for torus problems has recently been given (Cade 1978) in which the perturbation is of the integral equations of electrostatics. This is a more powerful theory, being applicable not only to interior problems, but to exterior problems, for which the methods of the present paper break down completely. However, the present theory has the advantage that, where it is applicable, it is easier. Thus, for example, whereas it would in principle be possible to obtain the fourth-order result (10) by the integral equation method, in practice an amount of labour comparable with that expended here would be required to go to the second order.

A virtue of the integral equation method is that two of the problems it solves are ones which have been solved exactly by other methods (the single circular torus and the flat ring, see Cade (1978)), for this gives us a clue regarding the accuracy we can expect from the perturbation theory generally. The indications are most encouraging. In the case of the single circular torus, the accuracy of the second-order capacity is 0·3% for values of a/b *going right up to*, although excluding, 1. In the case of the flat ring, the order of accuracy is even higher, 0·025%, which suggests that even where there is an error of 0·3%, one cannot be sure of where to place it (having regard for the fact that, with results based upon exact theory, difficult computations are involved). But most striking is the fact that, in both cases, the order of accuracy subsists for values up to (but excepting) unity of the perturbation parameter, something which we are not at all entitled to expect.

Evidently we can expect accuracy of this kind with the results of the present paper, except possibly where double perturbation analysis is involved, and we may speculate that the fourth-order result (10) should be very accurate indeed.

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